JOURNAL OF APPROXIMATION THEORY **90**, 46–74 (1997)
ARTICLE NO. AT963063

On Wavelets and Prewavelets with Vanishing Moments in Higher Dimensions

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Communicated by R. Q. Jia

Received June 26, 1995; accepted in revised form May 31, 1996

Using an approximation theory approach, we prove that a scaling function ϑ with suitable polynomial decay satisfies the Strang-Fix condition of order $r \in \mathbf{N}$ if and only if the elements of any prewavelet set $\{\psi_\nu\}_{\nu \in E^*}$ with polynomial decay of the same order have vanishing integral moments up to order $r-1$. An analogous equivalence is established that does not involve any assumptions concerning decay; this yields a new characterization of the rate of L^2-approximation of (stationary and nonstationary) multiresolution analyses in terms of a corresponding prewavelet set. Furthermore, we show that the existence of a scaling function with polynomial decay implies the existence of both an orthonormal scaling function and a wavelet set with polynomial decay of the same order. Several known constructions of wavelets and prewavelets are discussed in this respect.

1. INTRODUCTION

For many years authors have been studying the approximation properties of spaces generated by the integer translates of one or several given functions ([45, 16, 17, 8, 3, 34, 27, 4, 19], just to name a few) and the related problem of approximation by sampling series or, more generally, by quasi-interpolation operators ([42, 41, 13, 40, 20, 26, 11, 12, 22, 9, 21], among many others). An important role in this context plays the Strang-Fix condition: A function $\vartheta \in L^1(\mathbf{R}^d)$ satisfies the *Strang-Fix condition of order r* for some $r \in \mathbf{N}$ iff

$$D^\alpha \hat{\vartheta}(2j\pi) = 0, \qquad j \in \mathbf{Z}^d \setminus \{0\}, \qquad |\alpha| < r; \qquad \hat{\vartheta}(0) \neq 0. \tag{1.1}$$

To ensure that the Fourier transform $\hat{\vartheta}$ is $(r-1)$ times continuously differentiable, we will need a suitable decay condition upon ϑ. A sufficient condition would be to require $(1 + \|\cdot\|)^{r-1}\, \vartheta \in L^1(\mathbf{R}^d)$. In our setting it is

0021-9045/97 $25.00

useful to define the slightly more restrictive spaces $\mathscr{L}^p_{r-1}(\mathbf{R}^d)$: For $1 \leqslant p \leqslant \infty$ let $\mathscr{L}^p(\mathbf{R}^d)$ be the space introduced by R. Q. Jia and C. A. Micchelli in [28], which consists of all measurable functions f such that $\sum_{j \in \mathbf{Z}^d} |f(\cdot - j)| \in L^p([0,1]^d)$. Note that $\mathscr{L}^p(\mathbf{R}^d) \subset L^p(\mathbf{R}^d)$, $1 \leqslant p \leqslant \infty$, with $\mathscr{L}^1(\mathbf{R}^d) = L^1(\mathbf{R}^d)$ [28]. Further,

$$\mathscr{L}^p_r(\mathbf{R}^d) := \{ f \in L^1(\mathbf{R}^d);\, (1 + \|\cdot\|)^r f \in \mathscr{L}^p(\mathbf{R}^d) \}, \qquad 1 \leqslant p \leqslant \infty, \quad r \in \mathbf{N}_0, \tag{1.2}$$

see also [31]. If $1 \leqslant p \leqslant p' \leqslant \infty$ and r, $r' \in \mathbf{N}_0$ with $r \leqslant r'$, then $\mathscr{L}^{p'}_r(\mathbf{R}^d) \subset \mathscr{L}^p_r(\mathbf{R}^d)$ and $\mathscr{L}^p_{r'}(\mathbf{R}^d) \subset \mathscr{L}^p_r(\mathbf{R}^d)$. Obviously, $(1 + \|\cdot\|)^r f \in L^1(\mathbf{R}^d)$ for any function $f \in \mathscr{L}^p_r(\mathbf{R}^d)$, $1 \leqslant p \leqslant \infty$; in fact the elements of $\mathscr{L}^1_r(\mathbf{R}^d)$ are characterized by this property. Note that any function f satisfying $|f(x)| \leqslant K(1 + \|x\|)^{-r-d-\varepsilon}$, $x \in \mathbf{R}^d$, for some constant K and some $\varepsilon > 0$ is an element of $\mathscr{L}^p_r(\mathbf{R}^d)$. We will think of r as characterizing the polynomial decay rate of $f \in \mathscr{L}^p_r(\mathbf{R}^d)$.

In this paper we will only be concerned with $L^2(\mathbf{R}^d)$-functions, so we will choose our notations accordingly. The error of best approximation of a function $f \in L^2(\mathbf{R}^d)$ by a closed space $V \subset L^2(\mathbf{R}^d)$ will be denoted by

$$E(f, V) := \inf_{g \in V} \|f - g\|_2, \qquad f \in L^2(\mathbf{R}^d).$$

A closed space $V \subset L^2(\mathbf{R}^d)$ is called *shift-invariant*, if for any function $f \in V$ its *integer shifts* $f(\cdot - j)$, $j \in \mathbf{Z}^d$, are also contained in V. We say the shift-invariant space V is a *finitely generated shift-invariant space* (*FSI*), if there exists a finite set of functions $\{\vartheta_v\}_{v \in \mathscr{I}} \subset L^2(\mathbf{R}^d)$ such that V is the $L^2(\mathbf{R}^d)$-closure of span $\{\vartheta_v(\cdot - j);\, j \in \mathbf{Z}^d,\ v \in \mathscr{I}\}$. In case $\mathscr{I}$ consists of only one element v, V is called a *principal shift-invariant space* (*PSI*). We use the notation $V = S(\vartheta_v)$. As we will be interested in the approximation behaviour of the scaled versions of PSI's, it is convenient to use the notation

$$S^h(\vartheta) := \left\{ f\left(\frac{\cdot}{h}\right);\, f \in S(\vartheta) \right\}, \qquad h \in \mathbf{R}_+, \quad \vartheta \in L^2(\mathbf{R}^d).$$

Analogously, P^h_ϑ will denote the orthogonal projection to the space $S^h(\vartheta)$. A set of generators $\{\vartheta_v\}_{v \in \mathscr{I}}$ for an FSI V is called a *stable set of generators* if their integer shifts $\{\vartheta_v(\cdot - j)\}_{j \in \mathbf{Z}^d,\, v \in \mathscr{I}}$ form a Riesz-basis for V, see Section 2 for details. A special case is given by an *orthonormal set of generators*, when $\{\vartheta_v(\cdot - j)\}_{j \in \mathbf{Z}^d,\, v \in \mathscr{I}}$ forms an orthonormal basis for V.

Closely related to the theory of shift-invariant spaces is the notion of a *multiresolution analysis* (*MRA*) introduced by S. Mallat [36]. An MRA is a sequence $\{V_n\}_{n \in \mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R}^d)$ such that

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \tag{1.3}$$

$$\bigcap_{n \in \mathbf{Z}} V_n = \{0\}; \qquad \overline{\bigcup_{n \in \mathbf{Z}} V_n} = L^2(\mathbf{R}^d); \tag{1.4}$$

$$\textit{For all } n \in \mathbf{Z},\, f \in V_n \textit{ if and only if } f(2^{-n}\cdot) \in V_0; \tag{1.5}$$

$$V_0 \textit{ is a shift-invariant space}; \tag{1.6}$$

$$\textit{There exists a stable generator } \vartheta \textit{ for } V_0. \tag{1.7}$$

As C. de Boor, R. A. DeVore and A. Ron [6] show, the first condition in (1.4) is always satisfied in the presence of (1.3), (1.5)–(1.7), while the second is equivalent to $\bigcup_{n \in \mathbf{Z}} \operatorname{supp} \hat{\vartheta}(2^n \cdot) = \mathbf{R}^d$ up to a set of measure zero. Therefore, if $\hat{\vartheta}$ is continuous with $\hat{\vartheta}(0) \neq 0$, then (1.4) is satisfied automatically.

The function ϑ in (1.7) is called a *scaling function* for the MRA. Note that with the notation introduced above, $V_n = S^{2^{-n}}(\vartheta)$ for all $n \in \mathbf{Z}$. In the present situation, some authors prefer to use the term *stationary* multiresolution analysis, while a general (non-stationary) MRA is a sequence $\{V_n\}_{n \in \mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R}^d)$ satisfying (1.3) and $V_n = S^{2^{-n}}(\vartheta_n)$ for some $\vartheta_n \in L^2(\mathbf{R}^d)$ having stable integer shifts, $n \in \mathbf{N}$. So in this case we still require the space V_n to be a dyadic dilate of some PSI $S(\vartheta_n)$, but the generators on different levels may be different. Sometimes even the stability assumption is relaxed. Meaningful examples of a non-stationary MRA are provided by exponential B-splines; see [4, 6]. Our main interest is in the stationary case, so whenever we use the term "MRA" we mean a sequence of spaces satisfying (1.3)–(1.7).

Given an MRA we may introduce the "difference space" $W_n := V_{n+1} \ominus V_n$ as the orthogonal complement of V_n in V_{n+1}, $n \in \mathbf{Z}$. By (1.4)

$$L^2(\mathbf{R}^d) = V_n \oplus \bigoplus_{k \geqslant n} W_k = \bigoplus_{k \in \mathbf{Z}} W_k, \qquad n \in \mathbf{Z}.$$

It is easily checked that W_0 is a shift-invariant space. A stable set of generators for W_0 is called a *prewavelet set*, an orthonormal set of generators for W_0 will be called a *wavelet set*. It was shown in [6] that any prewavelet set necessarily consists of $2^d - 1$ elements. It will be convenient to index a prewavelet set by the set $E^* := E \backslash \{0\}$, where $E := \{0, 1\}^d$ is the set of vertices of the unit cube $[0, 1]^d$.

In the instance of an MRA one can either use properties of a scaling function or properties of a prewavelet set to characterize properties of the MRA. As we have seen above, when using the scaling function ϑ to describe the appoximation behaviour of the spaces $V_n = S^{2^{-n}}(\vartheta)$, $n \in \mathbf{Z}$, a suitable criterion is the Strang–Fix condition. As it turns out, a corresponding criterion when using a prewavelet set is the vanishing moment condition:

A finite set of functions $\{\vartheta_\nu\}_{\nu \in \mathcal{I}} \subset \mathcal{L}^1_{r-1}(\mathbf{R}^d)$ is said to satisfy the *vanishing moment condition of order* $r \in \mathbf{N}$ iff

$$D^\alpha \hat{\vartheta}_\nu(0) = 0, \qquad |\alpha| < r, \qquad \nu \in \mathcal{I}. \tag{1.8}$$

Note that this is equivalent to $\int_{\mathbf{R}^d} u^\alpha \vartheta_\nu(u)\, du = 0$, $|\alpha| < r$, $\nu \in \mathcal{I}$.

Wavelets or prewavelets satisfying the vanishing moment condition were used e.g. in [46, 21]to characterize the approximation behaviour of an MRA or associated operators; G. Beylkin, R. Coifman and V. Rokhlin [2] use wavelets with vanishing moments to compress large matrices.

Another concept that is widely used is regularity of a scaling function or a prewavelet set [36, 37, 1]. A function is said to be *r-regular* for some $r \in \mathbf{N}$, if it has rapid decay or at least polynomial decay of sufficiently high order and continuous derivatives up to order $r-1$ with the same decay properties. This concept is useful in that r-regularity of a scaling function implies that it satisfies the Strang-Fix condition of order r ([37, p. 56], [14, p. 158]), while r-regularity of a wavelet set implies that it satisfies the vanishing moment condition of order r ([37, p. 93], [18, p. 153]). Furthermore, one can show that the existence of an r-regular scaling function implies the existence of an r-regular wavelet set [24, 37]. So in the presence of an r-regular scaling function one can work interchangeably with the concept of a scaling function satisfying the Strang-Fix condition of order r or with the concept of a wavelet set satisfying the vanishing moment condition of order r. One purpose of this paper is to show that the two concepts are equivalent even when no regularity is present, assuming only sufficient decay of the functions involved to ensure that the corresponding Fourier transforms are smooth enough to give meaning to (1.1) and (1.8). Some of the implications involved have been treated by several authors under various assumptions [18, 36, 37, 44, 43]; a complete treatment of the univariate case is given in [10]. However, the method used there cannot be used in the general multivariate situation since it relies on an explicit representation of the wavelet of a specific form that is not known for the general case. See Section 7 for more details. We therefore choose to use an approximation theory approach.

The main result of this paper is the content of the following theorem.

THEOREM 1.1. *Let* $r \in \mathbf{N}$, *suppose* $\vartheta \in \mathcal{L}^\infty_r(\mathbf{R}^d)$ *is a scaling function for an MRA. The following are equivalent*:

$$\vartheta \text{ satisfies the Strang-Fix condition of order } r; \tag{1.9}$$

There exists an orthonormal scaling function $\varphi \in \mathscr{L}_r^\infty(\mathbf{R}^d)$ *satisfying the Strang-Fix condition of order* r; (1.10)

Any function $f \in V_0 \cap \mathscr{L}_r^\infty(\mathbf{R}^d)$ *satisfies*

$$D^\alpha \hat{f}(2j\pi) = 0, \qquad j \in \mathbf{Z}^d \backslash \{0\}, \qquad \alpha \in \mathbf{N}_0^d, \qquad |\alpha| < r; \tag{1.11}$$

There exists a prewavelet set $\{\omega_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$ *satisfying the vanishing moment condition of order* r; (1.12)

There exists a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$ *satisfying the vanishing moment condition of order* r; (1.13)

Any function $f \in W_0 \cap \mathscr{L}_r^\infty(\mathbf{R}^d)$ *satisfies*

$$D^\alpha \hat{f}(4j\pi) = 0, \qquad j \in \mathbf{Z}^d, \qquad \alpha \in \mathbf{N}_0^d, \qquad |\alpha| < r. \tag{1.14}$$

Note that we postulate the existence of a scaling function $\vartheta \in \mathscr{L}_r^\infty(\mathbf{R}^d)$, so that assertion (1.11) is not vacuous. We will show that the existence of a scaling function $\vartheta \in \mathscr{L}_r^\infty(\mathbf{R}^d)$ implies the existence of a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$; hence assertion (1.14) is not vacuous either and in fact trivially implies (1.13). We do not know at present whether the converse is true, i.e., whether the existence of a prewavelet set $\{\omega_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$ implies the existence of a scaling function $\vartheta \in \mathscr{L}_r^\infty(\mathbf{R}^d)$. Some results in this respect were obtained by P. G. Lemarié [32, 33]; he shows that in the univariate case the existence of a wavelet with compact support implies the existence of a scaling function with compact support. We do not know of any results concerning weaker decay properties, such as exponential or polynomial decay. Therefore to formulate a meaningful theorem we cannot renounce the a priori assumption $\vartheta \in \mathscr{L}_r^\infty(\mathbf{R}^d)$.

The proof of Theorem 1.1 will depend on the following theorem that will be established in Section 3 and is an interesting result in itself. To better understand its assumptions, note that for any function $\theta \in L^2(\mathbf{R}^d)$, the space $S^{1/2}(\theta)$ is an FSI generated by the functions $\{\theta(2\cdot - \mu)\}_{\mu \in E}$. As was shown in [5, Corollary 3.4 and Theorem 3.5], this implies that choosing an arbitrary function $\vartheta \in S^{1/2}(\theta)$, then $S^{1/2}(\theta)$ can be written as the orthogonal sum of $S(\vartheta)$ and $|E| = 2^d$ (or less) PSI's. Thus the assumption $S^{1/2}(\theta) \ominus S(\theta) = \bigoplus_{\nu \in \mathscr{I}} S(\omega_\nu)$ is made basically for notational reasons.

THEOREM 1.2. *Let* $r \in \mathbf{N}$; *suppose for* $\theta \in L^2(\mathbf{R}^d)$ *there holds* $\overline{\bigcup_{n \in \mathbf{Z}} S^{2^{-n}}(\theta)} = L^2(\mathbf{R}^d)$ *and* $S(\theta) \subset S^{1/2}(\theta)$. *Further suppose that the orthogonal complement of* $S(\theta)$ *in* $S^{1/2}(\theta)$ *is an FSI that has an orthogonal decomposition into finitely many PSI's*

$$S^{1/2}(\theta) \ominus S(\theta) = \bigoplus_{\nu \in \mathscr{I}} S(\omega_\nu),$$

with $\omega_\nu \in L^2(\mathbf{R}^d)$, $\nu \in \mathscr{I}$, $\mathscr{I}$ *a finite index set. The following conditions are equivalent*:

$$\|\cdot\|^{-r}\left(1-\frac{|\hat{\theta}|^2}{\sum_{j\in\mathbf{Z}^d}|\hat{\theta}(\cdot-2\pi j)|^2}\right)^{1/2}\in L^\infty([-\pi,\pi]^d); \tag{1.15}$$

There exists a constant $K>0$ *such that*

$$E(f, S^{2^{-n}}(\theta)) \leqslant K 2^{-nr}\,\|f\|_{r,2}, \qquad f\in\mathscr{W}_r^2(\mathbf{R}^d), \qquad n\in\mathbf{Z}; \tag{1.16}$$

There exists a constant $\tilde{K}>0$ *such that*

$$\|P_{\omega_\nu}^{2^{-n}} f\|_2 \leqslant \tilde{K} 2^{-nr}\,\|f\|_{r,2}, \qquad f\in\mathscr{W}_r^2(\mathbf{R}^d), \qquad n\in\mathbf{Z}, \qquad \nu\in\mathscr{I}; \tag{1.17}$$

$$\|\cdot\|^{-r}\frac{|\hat{\omega}_\nu|}{(\sum_{j\in\mathbf{Z}^d}|\hat{\omega}_\nu(\cdot-2\pi j)|^2)^{1/2}}\in L^\infty([-\pi,\pi]^d) \qquad \nu\in\mathscr{I}. \tag{1.18}$$

Condition (1.15) is closely related to the Strang-Fix condition but does not require differentiability of the Fourier transform; in fact, if $\hat{\theta}\in C_B^r(\mathbf{R}^d)$, then (1.15) implies the Strang-Fix condition of order r. Under additional assumptions (for example stability of shifts), the converse implication is true as well. The connections between (1.15) and the Strang-Fix condition are examined in detail in [4], see also Section 6. Similarly, condition (1.18) is closely related to the vanishing moment condition; more precisely, in case $\{\hat{\omega}_\nu\}_{\nu\in\mathscr{I}}\subset C_B^r(\mathbf{R}^d)$, the vanishing moment condition of order r is equivalent to (1.18), see Section 6. Thus one may consider the equivalence of (1.15) and (1.18) established in Theorem 1.2 as an extension of the equivalence of (1.9) and (1.12) in Theorem 1.1 to a more general setting where no stability or decay assumptions are made. Theorem 1.2 further establishes the connection between the generalized Strang-Fix condition (1.15), the generalized vanishing moment condition (1.18) and the rate of $L^2(\mathbf{R}^d)$-approximation of smooth functions by the ladder of spaces $\{S^{2^{-n}}(\theta)\}_{n\in\mathbf{Z}}$.

The equivalence of (1.15) and (1.16) was basically proved by C. de Boor, R. A. DeVore and A. Ron in [4]. More precisely they showed the equivalence of (1.15) with

$$E(f, S^h(\vartheta)) \leqslant K h^r\,\|f\|_{r,2}, \qquad f\in\mathscr{W}_r^2(\mathbf{R}^d), \quad h\in\mathbf{R}_+. \tag{1.19}$$

Choosing $h:=2^{-n}$, $n\in\mathbf{Z}$, then (1.19) turns into (1.16). No argument is used in their proof that would require a continuous parameter h.

The proof of the equivalence of (1.17) and (1.18) will proceed along the same lines; it will make use of several results established in [4]. Note that the denominator in (1.15) and (1.18) can only be zero if the numerator vanishes as well. As in [4], this possibility is taken care of by the convention that zero times any extended number is zero.

We would like to mention that similar characterizations of $L^\infty(\mathbf{R}^d)$-approximation by an MRA were obtained by S. E. Kelly, M. A. Kon and L. A. Raphael [29] in the context of "radially bounded" scaling functions and wavelets.

The paper is organized as follows: Section 2 is concerned with notations. In Section 3 a version of Theorem 1.2 for a non-stationary MRA is established and a proof of Theorem 1.2 is given. Section 4 deals with some basic facts about polynomial decay in an FSI; in particular we show that the existence of a stable set of generators $\{\theta_\nu\}_{\nu \in \mathscr{I}} \subset \mathscr{L}_r^p(\mathbf{R}^d)$ for some $2 \leqslant p \leqslant \infty$, $r \in \mathbf{N}_0$, implies the existence of an orthonormal set of generators with (at least) the same decay properties. In Section 5 we show that for any MRA associated with a scaling function $\vartheta \in \mathscr{L}_r^p(\mathbf{R}^d)$, $2 \leqslant p \leqslant \infty$, $r \in \mathbf{N}$, there also exists a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^p(\mathbf{R}^d)$. Several well-known constructions of wavelets are discussed in this respect. The proof of Theorem 1.1 is given in Section 6, while Section 7 contains some remarks concerning our decay assumptions.

2. SOME NOTATIONS

Throughout the paper we will use standard multiindex notation: The order of a multiindex $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{N}_0^d$ is given by $|\alpha| := \alpha_1 + \cdots + \alpha_d$, and for any two vectors $x, y \in \mathbf{R}^d$, $xy := x_1 y_1 + \cdots + x_d y_d$, and $\|x\| := (xx)^{1/2}$. Also $x^\alpha := \prod_{\nu=1}^d x_\nu^{\alpha_\nu}$.

$C_B(\mathbf{R}^d)$ is the space of all continuous and bounded functions on $\mathbf{R}^d$, endowed with the supremum norm $\|f\|_B := \sup_{x \in \mathbf{R}^d} |f(x)|$, and $C_{2\pi}$ is its subspace of 2π-periodic functions, i.e., functions $f \in C_B(\mathbf{R}^d)$ that are invariant under translation by elements of $2\pi\mathbf{Z}^d$. For $r \in \mathbf{N}$, $C_B^r(\mathbf{R}^d)$ is the space of all functions $f \in C_B(\mathbf{R}^d)$ which are r times differentiable on $\mathbf{R}^d$ with $D^\alpha f := (\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \cdots (\partial^{\alpha_d}/\partial x_d^{\alpha_d}) f \in C_B(\mathbf{R}^d)$ for all $\alpha \in \mathbf{N}_0$, $|\alpha| \leqslant r$, and $C_{2\pi}^r$ is its subspace of 2π-periodic functions.

For $1 \leqslant p < \infty$ and a countable set $\mathscr{I}$, $l^p(\mathscr{I})$ is the space of all complex-valued sequences over $\mathscr{I}$ such that $\|a\|_p := (\sum_{l \in \mathscr{I}} |a_l|^p)^{1/p} < \infty$, and $L^p(\mathbf{R}^d)$ is the space of measurable functions $f \colon \mathbf{R}^d \to \mathbf{C}$ with finite norm $\|f\|_p := (\int_{\mathbf{R}^d} |f(u)|^p \, du)^{1/p} < \infty$ (with the usual identifications). The Fourier transform of $f \in L^1(\mathbf{R}^d)$ is given by $\hat{f}(v) := (\sqrt{2\pi})^{-d} \int_{\mathbf{R}^d} f(u) \, e^{-ivu} \, du$, $v \in \mathbf{R}^d$, while for $f \in L^2(\mathbf{R}^d)$ it is defined by the $L^2(\mathbf{R}^d)$-limit function lim i.m.$_{M \to \infty} \widehat{(f \cdot \chi_{[-M, M]^d})}$, $\chi_{[-M, M]^d}$ being the characteristic function of the interval $[-M, M]^d$. In both cases the Fourier transform will be denoted by $\hat{f}$. For $r \in \mathbf{N}$, the Sobolev space $\mathscr{W}_2^r$ is the space of all functions

$f \in L^2(\mathbf{R}^d)$ with $\|f\|_{r,2} := \|(1+\|\cdot\|)^r \hat{f}\|_2 < \infty$. The scalar product of two functions $f, g \in L^2(\mathbf{R}^d)$ is given by $\langle f, g\rangle := \int_{\mathbf{R}^d} f(u)\,\overline{g(u)}\,du$.

For $1 \leqslant p < \infty$, $L^p_{2\pi}$ is the space of all 2π-periodic measurable functions f on $\mathbf{R}^d$ such that $\|f\|_p := (\int_{[-\pi,\pi]^d} |f(u)|^p\,du)^{1/p} < \infty$. The Fourier coefficients of $f \in L^p_{2\pi}$ are given by $\hat{f}(k) := (2\pi)^{-d} \int_{[-\pi,\pi]^d} f(u)\,e^{-iku}\,du$, $k \in \mathbf{Z}^d$.

A set of generators $\{\vartheta_\nu\}_{\nu\in\mathscr{I}}$ for an FSI V is called a stable set of generators if their integer shifts $\{\vartheta_\nu(\cdot - j)\}_{j\in\mathbf{Z}^d,\,\nu\in\mathscr{I}}$ form a Riesz-basis for V, i.e., if for any function $f \in V$ there exists a unique sequence $a = \{a_{j,\nu}\}_{j\in\mathbf{Z}^d,\,\nu\in\mathscr{I}} \in l^2(\mathbf{Z}^d \times \mathscr{I})$ such that f has a representation

$$f = \sum_{\nu\in\mathscr{I}} \sum_{j\in\mathbf{Z}^d} a_{j,\nu}\vartheta_\nu(\cdot - j),$$

and there exist constants $A, B > 0$ such that for any sequence $a \in l^2(\mathbf{Z}^d \times \mathscr{I})$,

$$A \sum_{\nu\in\mathscr{I}} \sum_{j\in\mathbf{Z}^d} |a_{j,\nu}|^2 \leqslant \left\| \sum_{\nu\in\mathscr{I}} \sum_{j\in\mathbf{Z}^d} a_{j,\nu}\vartheta_\nu(\cdot - j) \right\|_2^2 \leqslant B \sum_{\nu\in\mathscr{I}} \sum_{j\in\mathbf{Z}^d} |a_{j,\nu}|^2.$$

Thus, in this case

$$V = \left\{ f = \sum_{\nu\in\mathscr{I}} \sum_{j\in\mathbf{Z}^d} a_{j,\nu}\vartheta_\nu(\cdot - j);\ a = \{a_{j,\nu}\}_{j\in\mathbf{Z}^d,\,\nu\in\mathscr{I}} \in l^2(\mathbf{Z}^d \times \mathscr{I}) \right\}$$

or, taking Fourier transforms on both sides,

given a stable set of generators $\{\vartheta_\nu\}_{\nu\in\mathscr{I}}$ for a FSI V, then

$$V = \left\{ f \in L^2(\mathbf{R}^d);\ \hat{f} = \sum_{\nu\in\mathscr{I}} m_\nu \hat{\vartheta}_\nu \text{ for some } m_\nu \in L^2_{2\pi},\ \nu \in \mathscr{I} \right\}$$

(see [4]). We say a function $\vartheta \in L^2(\mathbf{R}^d)$ has *stable integer shifts* (or *orthonormal integer shifts*), if ϑ is a stable generator (or orthonormal generator) for the PSI $S(\vartheta)$. To get a handy characterization of these properties it is useful to introduce the *bracket product* as the 2π-periodic function

$$[\![f, g]\!] := (2\pi)^d \sum_{j\in\mathbf{Z}^d} \hat{f}(\cdot + 2\pi j)\,\overline{\hat{g}(\cdot + 2\pi j)}, \qquad f, g \in L^2(\mathbf{R}^d). \tag{2.2}$$

Then $[\![f, g]\!] \in L^1_{2\pi}$ for any two functions $f, g \in L^2(\mathbf{R}^d)$. By Poisson's summation formula, the corresponding Fourier series is given by

$$[\![f, g]\!] \sim \sum_{j\in\mathbf{Z}^d} \langle f, g(\cdot + j)\rangle\, e^{ij\cdot}.$$

If $f, g \in \mathscr{L}^2(\mathbf{R}^d)$, then the sequence of Fourier coefficients $\{\langle f, g(\cdot + j)\rangle\}_{j \in \mathbf{Z}^d}$ is in $l^1(\mathbf{Z}^d)$, the series in (2.2) converges pointwise, and there holds the pointwise equality

$$[\![f, g]\!](v) = \sum_{j \in \mathbf{Z}^d} \langle f, g(\cdot - j)\rangle\, e^{-ijv}, \qquad v \in \mathbf{R}^d, \qquad f, g \in \mathscr{L}^2(\mathbf{R}^d), \tag{2.3}$$

as was shown by R. Q. Jia and C. A. Micchelli [28]. (Note that our notation differs from the one used in [28] in that we choose to denote by $[\![f, g]\!]$ the trigonometric function associated with the sequence $\{\langle f, g(\cdot + j)\rangle\}_{j \in \mathbf{Z}^d}$ rather than the Laurent series.) It is easy to see that a function $f \in \mathscr{L}^2(\mathbf{R}^d)$ is orthogonal to $S(g)$ for $g \in \mathscr{L}^2(\mathbf{R}^d)$ if and only if $[\![f, g]\!] \equiv 0$ on $\mathbf{R}^d$. Also,

a function $\vartheta \in \mathscr{L}^2(\mathbf{R}^d)$ has orthonormal integer shifts if and only if $[\![\vartheta, \vartheta]\!](v) = 1$ for all $v \in \mathbf{R}^d$, (2.4)

and (see [28])

$\vartheta \in \mathscr{L}^2(\mathbf{R}^d)$ has stable integer shifts if and only if $[\![\vartheta, \vartheta]\!](v) > 0$ for all $v \in \mathbf{R}^d$. (2.5)

From (2.5) and the definition of the bracket product (2.2) it is easily seen that

a function $\vartheta \in \mathscr{L}^1_{r-1}(\mathbf{R}^d)$ having stable integer shifts satisfies the Strang-Fix condition of order r if and only if

$$D^\alpha \hat{\vartheta}(2j\pi) = 0, \qquad j \in \mathbf{Z}^d \backslash \{0\}, \qquad |\alpha| < r, \tag{2.6}$$

i.e., the assumption $\hat{\vartheta}(0) \neq 0$ in (1.1) can be dropped in this case.

3. RATES OF APPROXIMATION

In this section we will establish a version of Theorem 1.2 for the case of a non-stationary MRA and then give a proof of Theorem 1.2. This section may be thought of as an extension of some of the results established by C. de Boor, R. A. DeVore and A. Ron in [4]. While they give characterizations of the approximation order of stationary and non-stationary MRAs in terms of the generators, here a characterization in terms of "generalized prewavelets" is added.

THEOREM 3.1. *Let $r \in \mathbf{N}$, suppose for $\{\vartheta_n\}_{n \in \mathbf{Z}} \subset L^2(\mathbf{R}^d)$ there holds*

$$\overline{\bigcup_{n \in \mathbf{Z}} S^{2^{-n}}(\vartheta_n)} = L^2(\mathbf{R}^d) \tag{3.1}$$

and

$$S(\vartheta_n) \subset S^{1/2}(\vartheta_{n+1}), \qquad n \in \mathbf{Z}. \tag{3.2}$$

Further suppose that for any $n \in \mathbf{Z}$ *there exist functions* $\{\omega_{n,\nu}\}_{\nu \in \mathscr{I}} \subset L^2(\mathbf{R}^d)$ *such that the orthogonal complement* $S^{1/2}(\vartheta_{n+1}) \ominus S(\vartheta_n)$ *can be decomposed into an orthogonal sum of* $|\mathscr{I}|$ *PSI's*

$$S^{1/2}(\vartheta_{n+1}) \ominus S(\vartheta_n) = \bigoplus_{\nu \in \mathscr{I}} S(\omega_{n,\nu}), \tag{3.3}$$

where $\mathscr{I}$ *is assumed to be a finite index set independent of n. The following conditions are equivalent*:

$$\left\{(2^{-n} + \|\cdot\|)^{-r} \left(1 - \frac{(2\pi)^d\, |\hat{\vartheta}_n|^2}{[\![\vartheta_n, \vartheta_n]\!]}\right)^{1/2}\right\}_{n \in \mathbf{Z}}$$

is bounded in $L^\infty([-\pi, \pi]^d)$; (3.4)

There exists a constant $K > 0$ *such that*

$$E(f, S^{2^{-n}}(\vartheta_n)) \leqslant K 2^{-nr}\, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad n \in \mathbf{Z}; \tag{3.5}$$

There exists a constant $\tilde{K} > 0$ *such that*

$$\|P_{\omega_{n,\nu}}^{2^{-n}} f\|_2 \leqslant \tilde{K} 2^{-nr}\, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad n \in \mathbf{Z}, \quad \nu \in \mathscr{I}; \tag{3.6}$$

$$\left\{(2^{-n} + \|\cdot\|)^{-r} \frac{|\tilde{\omega}_{n,\nu}|}{[\![\omega_{n,\nu}, \omega_{n,\nu}]\!]^{1/2}}\right\}_{n \in \mathbf{Z}} \tag{3.7}$$

is bounded in $L^\infty([-\pi, \pi]^d)$ *for all* $\nu \in \mathscr{I}$.

As in Theorem 1.2, the assumption (3.3) is made basically for notational reasons. Concerning the equivalence of (3.4) and (3.5), let us quote the following result established in [4]:

Result 3.1 [4]. Let $r \in \mathbf{N}$, $\mathbf{A} \subset \mathbf{R}_+$ and $\{\vartheta_h\}_{h \in \mathbf{A}} \subset L^2(\mathbf{R}^d)$. There exists a constant $K > 0$ such that

$$E(f, S^h(\vartheta_h)) \leqslant K h^r\, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad h \in \mathbf{A},$$

if and only if

$$\left\{(h + \|\cdot\|)^{-r} \left(1 - \frac{(2\pi)^d\, |\hat{\vartheta}_h|^2}{[\![\vartheta_h, \vartheta_h]\!]}\right)^{1/2}\right\}_{h \in \mathbf{A}}$$

is bounded in $L^\infty([-\pi, \pi]^d)$.

Choosing $\mathbf{A} := \{2^{-n}; n \in \mathbf{Z}\}$, this obviously implies the equivalence of (3.4) and (3.5). Result 3.1 was actually established in [4] for the case $\mathbf{A} = \mathbf{R}_+$, but in the proof no argument is used that would require a continuous parameter.

To prove the equivalence of (3.6) and (3.7), we will establish a similar relation for the $\omega_{n,\nu}$, $n \in \mathbf{Z}$, $\nu \in \mathscr{I}$. The proof will be very similar to the proof of Result 3.1 as given in [4]; we will utilize several results established there. In particular, we will need the following fact which is one of the main ingredients of the proof of Result 3.1 in [4]:

Result 3.2 [4]. Let $r \in \mathbf{N}$, $\mathbf{A} \subset \mathbf{R}_+$, and $\{\Lambda_h\}_{h \in \mathbf{A}} \subset L^\infty([-\pi, \pi]^d)$. There exists a constant $K > 0$ such that

$$\left(\int_{[-\pi,\pi]^d/h} |\hat{f}|^2 \, |\Lambda_h(h\cdot)|^2\right)^{1/2} \leqslant Kh^r \, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad h \in \mathbf{A},$$

if and only if

$$\{(h + \|\cdot\|)^{-r} \, \Lambda_h\}_{h \in \mathbf{A}}$$

is bounded in $L^\infty([-\pi, \pi]^d)$.

Again, this fact is actually established for the case $\mathbf{A} = \mathbf{R}_+$.

THEOREM 3.2. *Let $r \in \mathbf{N}$, $\mathbf{A} \subset \mathbf{R}_+$ and $\{\omega_h\}_{h \in \mathbf{A}} \subset L^2(\mathbf{R}^d)$. There exists a constant $K > 0$ such that*

$$\|P^h_{\omega_h} f\|_2 \leqslant Kh^r \, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad h \in \mathbf{A}, \tag{3.8}$$

if and only if

$$\left\{(h + \|\cdot\|)^{-r} \frac{|\hat{\omega}_h|}{[\![\omega_h, \omega_h]\!]^{1/2}}\right\}_{h \in \mathbf{A}} \tag{3.9}$$

is bounded in $L^\infty([-\pi, \pi]^d)$.

Proof. First note that for fixed $h \in \mathbf{A}$, $|\hat{\omega}_h|/[\![\omega_h, \omega_h]\!]^{1/2}$ is bounded by $(2\pi)^{-d}$ by definition, therefore the expression in (3.9) is an element of $L^\infty([-\pi, \pi]^d)$ for any $h \in \mathbf{A} \subset \mathbf{R}_+$. Now let $f \in \mathscr{W}_r^2(\mathbf{R}^d)$, $h \in \mathbf{A}$. By a change of variables it is easily verified that

$$E(f, S^h(\omega_h)) = h^{d/2} E(f(h\cdot), S(\omega_h)), \tag{3.10}$$

hence (by orthogonality)

$$\begin{aligned}\|P^h_{\omega_h} f\|_2^2 &= \|f\|_2^2 - E(f, S^h(\omega_h))^2 \\ &= h^d(\|f(h\cdot)\|_2^2 - E(f(h\cdot), S(\omega_h))^2) \\ &= h^d \|P_{\omega_h} f(h\cdot)\|_2^2. \end{aligned} \tag{3.11}$$

Let $\check{\chi}_C$ be the inverse Fourier transform of the characteristic function of the cube $[-\pi, \pi]^d$, then

$$\begin{aligned} h^{d/2} \|P_{\omega_h}(f(h\cdot)) - P_{\omega_h}(f(h\cdot) * \check{\chi}_C)\|_2 &\leqslant h^{d/2} \|f(h\cdot)) - f(h\cdot) * \check{\chi}_C\|_2 \\ &= h^{d/2} \|(1-\chi_C)\, f(h\cdot)\,\hat{}\,\|_2. \end{aligned}$$

As was shown in [4], the last expression is bounded by

$$h^{d/2} \|(1-\chi_C)\, f(h\cdot)\,\hat{}\,\|_2 \leqslant \varepsilon_f(h)\, h^r \|f\|_{r,2},$$

where $\varepsilon_f(h): \mathbf{R}_+ \to \mathbf{R}$ is a nonnegative function satisfying $\varepsilon_f(h) \leqslant 1$ for all $h \in \mathbf{R}_+$ and $\lim_{h\to 0_+} \varepsilon_f(h) = 0$. Therefore (3.8) is equivalent to

$$h^{d/2} \|P_{\omega_h}(f(h\cdot) * \check{\chi}_C)\|_2 \leqslant K h^r \|f\|_{r,2}, \qquad f \in \mathscr{W}^2_r(\mathbf{R}^d), \quad h \in \mathbf{A}. \tag{3.12}$$

We need to establish the equivalence of (3.12) and (3.9). But $(f(h\cdot) * \check{\chi}_C)\,\hat{}\, = \chi_C f(h\cdot)\,\hat{}$ has support in $[-\pi, \pi]^d$; as was shown during the proof of Theorem 2.20 in [4], this implies that

$$\begin{aligned} &h^d \|P_{\omega_h}(f(h\cdot) * \check{\chi}_C)\|_2^2 \\ &\quad = h^d (2\pi)^d \int_{[-\pi,\pi]^d} |f(h\cdot)\,\hat{}\,(u)|^2\, |\hat{\omega}_h(u)|^2 / [\![\omega_h, \omega_h]\!](u)\, du \\ &\quad = h^{-d} (2\pi)^d \int_{[-\pi,\pi]^d} |\hat{f}(u/h)|^2\, |\hat{\omega}_h(u)|^2 / [\![\omega_h, \omega_h]\!](u)\, du \\ &\quad = (2\pi)^d \int_{[-\pi,\pi]^d/h} |\hat{f}(x)|^2\, |\hat{\omega}_h(hx)|^2 / [\![\omega_h, \omega_h]\!](hx)\, dx. \end{aligned}$$

Choosing $\Lambda_h := \hat{\omega}_h / [\![\omega_h, \omega_h]\!]^{1/2} \in L^\infty([-\pi, \pi]^d)$, $h \in \mathbf{A}$, the claim now follows from Result 3.2. ∎

Proof of Theorem 3.1. In view of Result 3.1 and Theorem 3.2 we only need to prove the equivalence of (3.5) and (3.6).

For $n \in \mathbf{Z}$, let $V_n := S^{2^{-n}}(\vartheta_n) = \{f(2^n\cdot);\, f \in S(\vartheta_n)\}$, and let

$$W_n := \{f(2^n\cdot);\, f \in S^{1/2}(\vartheta_{n+1}) \ominus S(\vartheta_n)\}$$

be the orthogonal complement of $S(\vartheta_n)$ in $S^{1/2}(\vartheta_{n+1})$, scaled by the factor 2^n. From (3.3) it follows that W_n can be written as the orthogonal sum

$$W_n = \bigoplus_{\nu \in \mathscr{I}} S^{2^{-n}}(\omega_{n,\nu}),$$

so in view of (3.1) we obtain the orthogonal decomposition

$$L^2(\mathbf{R}^d) = V_n \oplus \bigoplus_{k \geqslant n} W_k = V_n \oplus \bigoplus_{k \geqslant n} \left(\bigoplus_{\nu \in \mathscr{I}} S^{2^{-k}}(\omega_{k,\nu}) \right), \qquad n \in \mathbf{N}.$$

Thus, for any $n \in \mathbf{Z}$ and $f \in \mathscr{W}_r^2(\mathbf{R}^d)$,

$$E(f, S^{2^{-n}}(\vartheta_n))^2 = E(f, V_n)^2 = \sum_{k \geqslant n} \sum_{\nu \in \mathscr{I}} \|P^{2^{-k}}_{\omega_{k,\nu}} f\|_2^2.$$

In particular, $\|P^{2^{-n}}_{\omega_{n,\nu}} f\|_2 \leqslant E(f, S^{2^{-n}}(\vartheta_n))$ for all $n \in \mathbf{Z}$, $\nu \in \mathscr{I}$, so that (3.5) implies (3.6). On the other hand, if (3.6) is valid, then

$$E(f, S^{2^{-n}}(\vartheta_n))^2 \leqslant \tilde{K}^2 \|f\|_{r,2}^2 \sum_{k \geqslant n} \sum_{\nu \in \mathscr{I}} 2^{-2kr}$$

$$= \tilde{K}^2 \|f\|_{r,2}^2 \, 2^{-2nr} \sum_{k \geqslant n} \sum_{\nu \in \mathscr{I}} 2^{-2(k-n)r},$$

$$f \in \mathscr{W}_r^2(\mathbf{R}^d), \qquad n \in \mathbf{Z},$$

yielding (3.5), since $\mathscr{I}$ is finite. ∎

In case all the ϑ_n, $n \in \mathbf{Z}$, agree, Theorem 3.1 immediately yields Theorem 1.2:

Proof of Theorem 1.2. Choosing $\vartheta_n := \theta$, $n \in \mathbf{Z}$, and $\omega_{n,\nu} := \omega_\nu$, $n \in \mathbf{Z}$, $\nu \in \mathscr{I}$, in Theorem 3.1 and letting $n \to \infty$, we see that (3.4) is equivalent to (1.15), and (3.7) is equivalent to (1.18). ∎

Theorem 1.2 is in particular applicable when θ is a scaling function for an MRA. It will be needed in Section 6 in the instance of an orthonormal scaling function and a wavelet system:

COROLLARY 3.1. *Let $r \in \mathbf{N}$, suppose $\varphi \in L^2(\mathbf{R}^d)$ is an orthonormal scaling function for an MRA and $\{\psi_\nu\}_{\nu \in E^*} \subset L^2(\mathbf{R}^d)$ a corresponding wavelet set. The following conditions are equivalent*:

$$\|\cdot\|^{-r} (1 - (2\pi)^d |\hat{\varphi}|^2)^{1/2} \in L^\infty([-\pi, \pi]^d); \tag{3.13}$$

There exists a constant $K > 0$ such that

$$E(f, V_n) \leqslant K 2^{-nr} \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \qquad n \in \mathbf{Z}; \tag{3.14}$$

There exists a constant $\tilde{K} > 0$ *such that*

$$\|Q_n f\|_2 \leqslant \tilde{K} 2^{-nr} \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \qquad n \in \mathbf{Z}, \tag{3.15}$$

where Q_n *is the orthogonal projection to* W_n;

$$\|\cdot\|^{-r} |\widehat{\psi_v}| \in L^\infty([-\pi, \pi]^d), \qquad v \in E^*. \tag{3.16}$$

Proof. Identifying $\theta = \varphi$, $V_n = S^{2^{-n}}(\theta)$, $\omega_v = \psi_v$, $v \in E^*$, $\|Q_k f\|_2^2 = \sum_{v \in E^*} \|P_{\psi_v}^{2^{-k}} f\|_2^2$, $k \in \mathbf{Z}$, the proof of Corollary 3.1 follows immediately from Theorem 1.2 and (2.4). ▮

4. POLYNOMIAL DECAY IN AN FSI

Theorem 1.1 also involves assertions about polynomial decay properties of scaling functions and wavelets. Some of these will be dealt with in this section. Suppose $r \in \mathbf{N}_0$. Following J. Lei [30], let us introduce the commutative Banach-Algebra

$$A_r := \left\{ h \in C_{2\pi};\ \|h\|_{A_r} := \sum_{k \in \mathbf{Z}^d} (1 + \|k\|)^r\, |\hat{h}(k)| < \infty \right\}. \tag{4.1}$$

With reference to [23], J. Lei [30] shows that the only algebra homomorphisms mapping A_r to the complex plane $\mathbf{C}$ are the point functionals, hence the spectrum of a function $h \in A_r$ agrees with its range. From [23, p. 48, Theorem 1] he concludes:

For any function F analytic on a neighbourhood of the range of $h \in A_r$ there holds $F \circ h \in A_r$, where $\circ$ denotes composition of functions. (4.2)

The main result of this section is given by the following theorem.

THEOREM 4.1. *Let* $r \in \mathbf{N}_0$, $2 \leqslant p \leqslant \infty$; *suppose* $\{\theta_v\}_{v \in \mathscr{I}} \subset \mathscr{L}_r^p(\mathbf{R}^d)$ *is a stable set of generators for the FSI V. There exists an orthonormal set of generators* $\{\varphi_v\}_{v \in \mathscr{I}} \subset \mathscr{L}_r^p(\mathbf{R}^d)$ *with*

$$\hat{\varphi}_v = \sum_{\mu \in \mathscr{I}} \xi_{\mu,v} \hat{\theta}_\mu, \qquad v \in \mathscr{I}, \tag{4.3}$$

for some $\{\xi_{\mu,v}\}_{\mu,v \in \mathscr{I}} \subset A_r$.

This theorem was proved for the case $r = 0$, $p = \infty$ by R.-Q. Jia and C. A. Micchelli [28]; they also showed that from a stable set of generators with exponential decay one can derive an orthonormal set of generators with exponential decay. Polynomial decay of order $r \in \mathbf{N}$ was considered by J. Lei [30] in case of a PSI with $p = \infty$.

LEMMA 4.1. (a) *If* $f, g \in \mathscr{L}_r^p(\mathbf{R}^d)$ *for some* $2 \leqslant p \leqslant \infty$, $r \in \mathbf{N}_0$, *then* $[\![f, g]\!] \in A_r$.

(b) *Let* $f \in \mathscr{L}_r^p(\mathbf{R}^d)$ *and* $h \in A_r$ *for some* $1 \leqslant p \leqslant \infty$, $r \in \mathbf{N}_0$; *define* $g \in L^2(\mathbf{R}^d)$ *by* $\hat{g} := \hat{f} \cdot h$. *Then* $g \in \mathscr{L}_r^p(\mathbf{R}^d)$.

Proof. Concerning (a), for $k \in \mathbf{Z}^d$ arbitrary let

$$K_1(k) := \{x \in \mathbf{R}^d;\ \|x + k\| \geqslant \|k\|/2\};$$
$$K_2(k) := \{x \in \mathbf{R}^d;\ \|x\| \geqslant \|k\|/2\}.$$

Then $K_1(k) \cup K_2(k) = \mathbf{R}^d$, since $\|k\| \leqslant \|x + k\| + \|x\|$. Let $1 \leqslant q \leqslant 2$ with $1/p + 1/q = 1$. Then $q \leqslant p$, so $g \in \mathscr{L}_r^p(\mathbf{R}^d) \subset \mathscr{L}_r^q(\mathbf{R}^d)$, and $f, g \in \mathscr{L}_r^p(\mathbf{R}^d) \subset \mathscr{L}^p(\mathbf{R}^d) \subset \mathscr{L}^2(\mathbf{R}^d)$. By (2.3)

$$\begin{aligned}
&\sum_{k \in \mathbf{Z}^d} (1 + \|k\|)^r \, |[\![f, g]\!]\,\hat{}\,(k)| \\
&\quad = \sum_{k \in \mathbf{Z}^d} (1 + \|k\|)^r \, |\langle f, g(\cdot + k)\rangle| \\
&\quad \leqslant \sum_{k \in \mathbf{Z}^d} (1 + \|k\|)^r \left(\int_{K_1(k)} \frac{2(1 + \|x + k\|)^r}{(1 + \|k\|)^r} \, |f(x)\, g(x + k)| \, dx \right. \\
&\qquad \left. + \int_{K_2(k)} \frac{2(1 + \|x\|)^r}{(1 + \|k\|)^r} \, |f(x)\, g(x + k)| \, dx \right) \\
&\quad \leqslant 2 \sum_{k \in \mathbf{Z}^d} \int_{\mathbf{R}^d} (1 + \|x + k\|)^r \, |f(x)\, g(x + k)| \, dx \\
&\qquad + 2 \sum_{k \in \mathbf{Z}^d} \int_{\mathbf{R}^d} (1 + \|x - k\|)^r \, |f(x - k)\, g(x)| \, dx \\
&\quad = 2 \int_{[0,1]^d} \sum_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} (1 + \|x + k - j\|)^r \, |f(x - j)\, g(x + k - j)| \, dx \\
&\qquad + 2 \int_{[0,1]^d} \sum_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} (1 + \|x - k - j\|)^r \, |f(x - k - j)\, g(x - j)| \, dx \\
&\quad \leqslant 2 \left\| \sum_{k \in \mathbf{Z}^d} (1 + \|\cdot + k\|)^r \, |g(\cdot + k)| \right\|_q \left\| \sum_{j \in \mathbf{Z}^d} |f(\cdot - j)| \right\|_p \\
&\qquad + 2 \left\| \sum_{k \in \mathbf{Z}^d} (1 + \|\cdot - k\|)^r \, |f(\cdot - k)| \right\|_p \left\| \sum_{j \in \mathbf{Z}^d} |g(\cdot - j)| \right\|_q < \infty,
\end{aligned}$$

by Hölder's inequality.

Concerning (b), by the Fourier inversion theorem there holds

$$\hat{g}(v) = h(v)\,\hat{f}(v) = \sum_{k \in \mathbf{Z}^d} \hat{h}(k)\,e^{ikv}\hat{f}(v), \qquad v \in \mathbf{R}^d,$$

or

$$g = \sum_{k \in \mathbf{Z}^d} \hat{h}(k)\,f(\cdot + k).$$

Therefore

$$\begin{aligned}
&\sum_{j \in \mathbf{Z}^d} (1 + \|x-j\|)^r\,|g(x-j)| \\
&\quad = \sum_{j \in \mathbf{Z}^d} (1 + \|x-j\|)^r \left| \sum_{k \in \mathbf{Z}^d} \hat{h}(k)\,f(x-j+k \right| \\
&\quad \leqslant \sum_{j \in \mathbf{Z}^d} \sum_{k \in \mathbf{Z}^d} \frac{(1 + \|x-j\|)^r}{(1 + \|x-j+k\|)^r\,(1 + \|k\|)^r} \\
&\qquad \times (1 + \|k\|)^r\,|\hat{h}(k)|\,(1 + \|x-j+k\|)^r\,|f(x-j+k)| \\
&\quad \leqslant \|h\|_{A_r} \sum_{j \in \mathbf{Z}^d} (1 + \|x-j\|)^r\,|f(x-j)| \qquad \text{a.e. on } \mathbf{R}^d,
\end{aligned}$$

since $1 + \|x-j\| \leqslant (1 + \|x-j+k\|)(1 + \|k\|)$, $x \in \mathbf{R}^d$, $k \in \mathbf{Z}$. The right hand-side, however, is in $L^p([0,1]^d)$ by assumption, thus $g \in \mathscr{L}^p_r(\mathbf{R}^d)$. ∎

Proof of Theorem 4.1. Without loss of generality assume $\mathscr{I} = \{1, ..., |\mathscr{I}|\}$. R.-Q. Jia and C. A. Micchelli [28] show that V has an orthogonal decomposition into $|\mathscr{I}|$ PSI's, i.e.,

$$V = \bigoplus_{v \in \mathscr{I}} S(\varphi'_v), \tag{4.4}$$

where the functions φ'_v are defined iteratively by

$$\begin{aligned}
&\varphi'_1 := \theta_1; \\
&\hat{\varphi}'_{n+1} := \hat{\theta}_{n+1} + \sum_{v=1}^{n} \frac{[\![\theta_{n+1}, \varphi'_v]\!]}{[\![\varphi'_v, \varphi'_v]\!]}\,\widehat{\varphi'_v}, \qquad 1 \leqslant n < |\mathscr{I}|.
\end{aligned} \tag{4.5}$$

Moreover, the functions φ'_v have stable integer shifts for all $v \in \mathscr{I}$.

Let us show that $\{\varphi'_v\}_{v \in \mathscr{I}} \subset \mathscr{L}^p_r(\mathbf{R}^d)$. Suppose the claim has been proved for $\{\varphi'_v\}_{v=1}^{n}$ for some $n < |\mathscr{I}|$. Fix $v \in \{1, ..., n\}$. Then by Lemma 4.1(a),

$[\![\varphi'_\nu, \varphi'_\nu]\!] \in A_r$. Furthermore, $[\![\varphi'_\nu, \varphi'_\nu]\!](x) \neq 0$ for all $x \in \mathbf{R}^d$ by (2.5). From (4.2) it follows that $[\![\varphi'_\nu, \varphi'_\nu]\!]^{-1} \in A_r$ as well. Again by Lemma 4.1(a), $[\![\theta_{n+1}, \varphi'_\nu]\!] \in A_r$, hence $[\![\theta_{n+1}, \varphi'_\nu]\!]/[\![\varphi'_\nu, \varphi'_\nu]\!] \in A_r$. Thus $\varphi'_{n+1} \in \mathscr{L}^p_r(\mathbf{R}^d)$ by Lemma 4.1(b).

Note that by induction there exist $\xi'_{\mu,\nu} \in A_r$ such that

$$\hat{\varphi}'_\nu = \sum_{\mu \in \mathscr{I}} \xi'_{\mu,\nu} \hat{\theta}_\mu, \qquad \nu \in \mathscr{I}. \tag{4.6}$$

Finally, setting

$$\hat{\varphi}_\nu := \frac{\hat{\varphi}'_\nu}{[\![\varphi'_\nu, \varphi'_\nu]\!]^{1/2}}, \qquad \nu \in \mathscr{I},$$

then $[\![\varphi_\nu, \varphi_\nu]\!] = 1$, $\nu \in \mathscr{I}$; therefore by (2.4) the φ_ν have orthonormal integer shifts. By Lemma 4.1, $[\![\varphi'_\nu, \varphi'_\nu]\!] \in A_r \subset C_{2\pi}$, so from (2.1) and (2.5) we have $S(\varphi_\nu) = S(\varphi'_\nu)$, $\nu \in \mathscr{I}$. From (4.4) we conclude that $\{\varphi_\nu\}_{\nu \in \mathscr{I}}$ is an orthonormal set of generators for V. By the same argument as above $[\![\varphi'_\nu, \varphi'_\nu]\!]^{-1/2} \in A_r$; hence $\varphi_\nu \in \mathscr{L}^p_r(\mathbf{R}^d)$ and by (4.6) $\hat{\varphi}_\nu = \sum_{\mu \in \mathscr{I}} \xi_{\mu,\nu} \hat{\theta}_\mu$ with $\xi_{\mu,\nu} := \xi'_{\mu,\nu}/[\![\varphi'_\nu, \varphi'_\nu]\!]^{1/2} \in A_r$, $\mu, \nu \in \mathscr{I}$. ∎

5. SCALING FUNCTIONS AND WAVELETS WITH POLYNOMIAL DECAY

In this section we will show that for any MRA associated with a scaling function $\theta \in \mathscr{L}^p_r(\mathbf{R}^d)$, $2 \leqslant p \leqslant \infty$, there exists a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}^p_r(\mathbf{R}^d)$.

Let $\theta \in \mathscr{L}^p_r(\mathbf{R}^d)$, $2 \leqslant p \leqslant \infty$, be a scaling function for an MRA $\{V_n\}_{n \in \mathbf{Z}}$. By Theorem 4.1 (for the PSI case) there also exists an orthonormal scaling function $\varphi \in \mathscr{L}^p_r(\mathbf{R}^d)$. We are trying to find functions $\psi_\nu \in V_1 \cap \mathscr{L}^p_r(\mathbf{R}^d)$, $\nu \in E^*$, such that $\{\varphi\} \cup \{\psi_\nu\}_{\nu \in E^*}$ is an orthonormal set of generators for the shift-invariant space V_1. Let us for simplicity write

$$\psi_0 := \varphi. \tag{5.1}$$

Since the half-shifts of $2^{d/2}\varphi(2\,\cdot)$ provide an orthonormal basis for V_1, for any $\nu \in E$ there exists a sequence $\{a^\nu_j\}_{j \in \mathbf{Z}^d} \in l^2(\mathbf{Z}^d)$ such that

$$\psi_\nu = \sum_{j \in \mathbf{Z}^d} a^\nu_j \varphi(2 \cdot - j) \tag{5.2}$$

or, equivalently,

$$\hat{\psi}_\nu = \sum_{j \in \mathbf{Z}^d} a_j^\nu \, e^{-ij\cdot/2} 2^{-d} \hat{\varphi}(\cdot/2) = 2^{-d} q^\nu(\cdot/2) \, \hat{\varphi}(\cdot/2), \qquad \nu \in E, \tag{5.3}$$

with

$$q^\nu := \sum_{j \in \mathbf{Z}^d} a_j^\nu \, e^{-ij\cdot} \in L^2_{2\pi}, \qquad \nu \in E. \tag{5.4}$$

This series can be rewritten in the following way:

$$q^\nu(\cdot/2) = \sum_{\mu \in E} e^{-i\mu\cdot/2} \left(\sum_{j \in \mathbf{Z}^d} a^\nu_{\mu+2j} \, e^{-ij\cdot} \right) = \sum_{\mu \in E} e^{-i\mu\cdot/2} p^\nu_\mu, \qquad \mu, \nu \in E, \tag{5.5}$$

where

$$p^\nu_\mu := \sum_{j \in \mathbf{Z}^d} a^\nu_{\mu+2j} \, e^{-ij\cdot} \in L^2_{2\pi}, \qquad \mu, \nu \in E. \tag{5.6}$$

Thus (5.3) turns into

$$\hat{\psi}_\nu = 2^{-d} \sum_{\mu \in E} e^{-i\mu\cdot/2} p^\nu_\mu \hat{\varphi}(\cdot/2), \qquad \nu \in E. \tag{5.7}$$

For later use let us note that since $\varphi \in \mathscr{L}^p_r(\mathbf{R}^d)$ is an orthonormal scaling function, by (2.4) this implies for all $v \in \mathbf{R}^d$

$$\begin{aligned}
1 &= [\![\varphi, \varphi]\!](v) \\
&= (2\pi)^d \sum_{\mu \in E} \sum_{k \in \mathbf{Z}^d} |\hat{\varphi}(v - 2\mu\pi - 4k\pi)|^2 \\
&= \sum_{\mu \in E} 2^{-2d} \left| \sum_{\mu' \in E} e^{-i\mu'(v-2\mu\pi)/2} p^0_{\mu'}(v) \right|^2 (2\pi)^d \sum_{k \in \mathbf{Z}^d} |\hat{\varphi}(v/2 - \mu\pi - 2k\pi)|^2 \\
&= 2^{-2d} \sum_{\mu', \mu^* \in E} p^0_{\mu'}(v) \, \overline{p^0_{\mu^*}(v)} \, e^{i(\mu^* - \mu') v/2} \sum_{\mu \in E} (-1)^{(\mu' + \mu^*)\mu} \\
&= 2^{-d} \sum_{\mu' \in E} |p^0_{\mu'}(v)|^2.
\end{aligned}$$

Here to get from line 2 to line 3 we have used (2.4), while the last equation is a consequence of $\sum_{\mu \in E} (-1)^{(\mu' + \mu^*)\mu} = 2^d \delta_{\mu', \mu^*}$. Thus,

$$\sum_{\mu \in E} 2^{-d} \, |p^0_\mu(v)|^2 = 1, \qquad v \in \mathbf{R}^d. \tag{5.8}$$

Furthermore, since $\{2^{d/2}\varphi(2\cdot - j)\}_{j\in\mathbf{Z}}$ is an orthonormal basis for V_1, the coefficients in (5.2) are given by

$$a_j^\nu = 2^d\langle\psi_\nu, \varphi(2\cdot - j)\rangle, \qquad \nu\in E, \quad j\in\mathbf{Z}^d,$$

so we can write alternatively

$$\begin{aligned} p_\mu^\nu &= \sum_{j\in\mathbf{Z}^d} 2^d\langle\psi_\nu, \varphi(2\cdot - 2j - \mu)\rangle\, e^{-ij\cdot} \\ &= 2^d[\![\psi_\nu, \varphi(2\cdot - \mu)]\!], \qquad \mu, \nu\in E. \end{aligned} \tag{5.9}$$

It is easily verified that $\varphi\in\mathscr{L}_r^p(\mathbf{R}^d)$ implies $\varphi(2\cdot)\in\mathscr{L}_r^p(\mathbf{R}^d)$. Therefore Lemma 4.1 together with (5.7) and (5.9) yield $\psi_\nu\in\mathscr{L}_r^p(\mathbf{R}^d)$ if and only if $p_\mu^\nu\in A_r$, $\mu, \nu\in E$. In particular, by the choice (5.1) of ψ_0,

$$p_\mu^0 = 2^d[\![\varphi, \varphi(2\cdot - \mu)]\!]\in A_r, \qquad \mu\in E. \tag{5.10}$$

On the other hand, R.-Q. Jia and C. A. Micchelli [28, Theorem 7.1] show that $\{\psi_\nu\}_{\nu\in E}$ is an orthonormal generating set for V_1 if and only if the matrix $(2^{-d/2}p_\mu^\nu)_{\nu,\mu\in E}$ is unitary almost everywhere on $[-\pi, \pi]^d$. Thus our problem is reduced to finding functions $p_\mu^\nu\in A_r$, $\mu\in E$, $\nu\in E^*$, such that the matrix $(2^{-d/2}p_\mu^\nu)_{\nu,\mu\in E}$ is unitary a.e. (hence everywhere by continuity) on $\mathbf{R}^d$, where the first column is determined by (5.10). By (5.8), the problem is meaningful. In dimension $d=1$, $E=\{0, 1\}$, so we only have to choose p_0^1 and p_1^1. The standard choice is $p_0^1 = -\overline{p_1^0}$, $p_1^1 = \overline{p_0^0}$. The resulting matrix is unitary by (5.8); moreover (5.10) implies p_0^1, $p_1^1\in A_r$, so the problem is solved. Note that in view of (5.5), this choice leads to

$$q^1(v) = e^{-iv}\overline{q^0(v+\pi)}, \qquad v\in\mathbf{R}. \tag{5.11}$$

In higher dimensions the situation is more complicated. S. D. Riemenschneider and Z. Shen [38, 39] show that in certain instances a construction similar to the univariate case is possible: Let $\eta\colon E\mapsto E$ be a permutation satisfying

$$\eta(0)=0 \textit{ and } (\eta(\nu)+\eta(\mu))(\nu+\mu) \textit{ is odd for any } \nu, \mu\in E \textit{ with } \nu\neq\mu. \tag{5.12}$$

If the orthonormal scaling function φ is skew symmmetric about $c_\varphi\in 1/2\mathbf{Z}^d$, i.e.,

$$\varphi(c_\varphi + x) = \overline{\varphi(c_\varphi - x)}, \qquad x\in\mathbf{R}^d,$$

then the choice

$$q^{\nu} := e^{i\eta(\nu)\cdot} \begin{cases} q^0(\cdot + \pi\nu), & \text{if } 2\nu c_{\varphi} \text{ is even,} \\ \overline{q^0(\cdot + \pi\nu)}, & \text{if } 2\nu c_{\varphi} \text{ is odd,} \end{cases} \qquad \nu \in E^*, \tag{5.13}$$

yields a wavelet set for the corresponding MRA. This corresponds to the choice

$$p^{\nu}_{[\eta(\nu)+\mu]} = (-1)^{\mu\nu} \begin{cases} e^{i/2\{(\eta(\nu)-\mu)+[\eta(\nu)-\mu]\}\cdot} p^0_{\mu} \\ \qquad \text{if } 2\nu c_{\varphi} \text{ is even,} \\ e^{i/2\{(\eta(\nu)+\mu)+[\eta(\nu)-\mu]\}\cdot} \overline{p^0_{\mu}} \\ \qquad \text{if } 2\nu c_{\varphi} \text{ is odd,} \end{cases} \qquad \mu \in E, \quad \nu \in E^*,$$

where for $\alpha \in \mathbf{Z}^d$ we denote by $[\alpha]$ the representative of $\alpha \bmod 2\mathbf{Z}^d$ lying in E. (In particular, $\alpha - [\alpha] \in 2\mathbf{Z}^d$ for all $\alpha \in \mathbf{Z}^d$.) Combining this with (5.10) we find $p^{\nu}_{\mu} \in A_r$ for all $\mu \in E$, $\nu \in E^*$, so these wavelets also possess the desired decay properties. This construction was originally devised to derive compactly supported wavelets from a compactly supported orthonormal scaling function; S. D. Riemenschneider and Z. Shen [38] show that in addition these wavelets are skew-symmetric or skew-antisymmetric, an important property in filtering theory.

As S. D. Riemenschneider and Z. Shen [38] show, the assumption $c_{\varphi} \in 1/2\mathbf{Z}^d$ is not restrictive, since the center of any skew-symmetric scaling function must necessarily be a half-integer. In [38, 39] they give functions η satisfying (5.12) for dimension $d \leqslant 3$. In case $d = 1$ and $2c_{\varphi}$ odd, choosing $\eta(0) = 0$ and $\eta(1) = 1$, their construction (5.13) obviously agrees with the univariate one (5.11) considered above up to a translation. They also show that for $d > 3$ such a permutation η does not exist, so this construction is only possible in low dimensions. For arbitrary dimension d, let us therefore follow the more abstract approach which was used by K. Gröchenig [24] to show that the existence of an r-regular scaling function implies the existence of an r-regular wavelet set; see also [37, p. 90] and [43].

Consider the map $P\colon [-\pi, \pi]^d \to \mathbf{C}^{2^d}$ defined by

$$P_{\mu} := 2^{-d/2} p^0_{\mu}, \qquad \mu \in E.$$

By (5.8), P can actually be considered as a map to the (real) unit sphere $S^{2^{d+1}-1} \subset \mathbf{R}^{2^{d+1}}$. Now assume $r \geqslant 1$, then P is continuously differentiable and $d < 2^{d+1} - 1$; therefore the image

$$K := P([-\pi, \pi]^d)$$

has $(2^{d+1}-1)$-measure zero, hence cannot be the whole of $S^{2^{d+1}-1}$ (see e.g. [25, pp. 68/69]). Up to a rotation we may assume $(-1, 0, ..., 0) \notin K$. Since K is compact, there exists some $0 < s < 1$ such that $\Re e\, p_0^0(v) \geq -s$ for all $v \in [-\pi, \pi]^d$. For some $\varepsilon > 0$, consider the matrix

$$\tilde{\mathscr{P}} = (\tilde{p}_\mu^\nu)_{\mu, \nu \in E} := \begin{pmatrix} P_0 & -\bar{P}_{\mu_1} & \cdots & -\bar{P}_{\mu_{2^d-1}} \\ P_{\mu_1} & \varepsilon & & \mathbf{0} \\ \vdots & & \ddots & \\ P_{\mu_{2^d-1}} & \mathbf{0} & & \varepsilon \end{pmatrix}$$

(with some ordering $\mu_0 = 0, \mu_1, ..., \mu_{2^d-1}$ of E). This matrix has determinant

$$\det \tilde{\mathscr{P}} = \varepsilon^{2^d-2} \left(\varepsilon P_0 + \sum_{\mu \in E^*} |P_\mu|^2 \right).$$

This expression can only vanish in a point v where $P_0(v)$ is real and non-positive. In such a point, however, with (5.8) and the definition of P,

$$\varepsilon P_0(v) + \sum_{\mu \in E^*} |P_\mu(v)|^2 = \varepsilon P_0(v) + 1 - |P_0(v)|^2 \geq -\varepsilon s + 1 - s^2,$$

which is non-zero for $0 < \varepsilon < 1/s - s$. Using Gram–Schmidt orthogonalization without changing the first column results in a unitary matrix

$$\mathscr{P} = 2^{-d/2} (p_\mu^\nu)_{\mu, \nu \in E},$$

whose every entry can be written in the form $2^{-d/2} p_\mu^\nu = F_\mu^\nu \circ P$, where F_μ^ν is a complex valued function real analytic on an open neighbourhood U of K (considered here as a subset of $\mathbf{C}^{2^d}$), i.e., locally expandable into a power series in $(z_1, \bar{z}_1, z_2, \bar{z}_2, ..., z_{2^d}, \bar{z}_{2^d})$ on U. Our problem is solved if we can show that $p_\mu^\nu \in A_r$, $\mu, \nu \in E$. This, however, is an immediate consequence of the following lemma.

LEMMA 5.1. *Let $r \in \mathbf{N}_0$, $\{h_\mu\}_{\mu \in E} \subset A_r$; define $H: \mathbf{R}^d \to \mathbf{C}^{2^d}$ by $H_\mu := h_\mu$, $\mu \in E$, and let $C := H(\mathbf{R}^d) = H([-\pi, \pi]^d)$ be the common spectrum of the functions h_μ, $\mu \in E$. Further let f be a complex-valued function real analytic on an open neighbourhood of C. Then $f \circ H \in A_r$.*

The proof follows directly from [23, Sect. 13, Satz 1, p. 100], taking into account that A_r is closed under complex conjugation and that the only algebra homomorphisms mapping A_r to the complex plane $\mathbf{C}$ are the point functionals.

We have proved the following:

THEOREM 5.1. *Let $r \in \mathbf{N}$, $2 \leqslant p \leqslant \infty$, and $\vartheta \in \mathscr{L}_r^p(\mathbf{R}^d)$ be a scaling function for an MRA. Then there exists a wavelet set $\{\psi_v\}_{v \in E^*} \subset \mathscr{L}_r^p(\mathbf{R}^d)$. If dimension $d=1$ or if $d \leqslant 3$ and ϑ is a skew-symmetric orthonormal scaling function, then we can also allow $r=0$.*

Let us discuss some alternative possibilities for the construction of a wavelet set in $\mathscr{L}_r^p(\mathbf{R}^d)$. J. Stöckler [43] shows that in case $\hat{\varphi}$ is real and nonnegative, then p_0^0 is strictly positive as well and the above construction of wavelets can be based upon the matrix

$$\tilde{\mathscr{P}}(\tilde{p}_\mu^v)_{\mu, v \in E} := 2^{-d/2} \begin{pmatrix} p_0^0 & 0 & \cdots & 0 \\ p_{\mu_1}^0 & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ p_{\mu_{2^d-1}}^0 & \mathbf{0} & & 1 \end{pmatrix} \tag{5.15}$$

instead of (5.14). By the same argument as above, this construction will lead to a wavelet set $\{\psi_v\}_{v \in E^*} \in \mathscr{L}_r^p(\mathbf{R}^d)$. Another possibility to find a unitary matrix $\mathscr{P}$ having $2^{-d/2}(p_0^0, p_{\mu_1}^0, ..., p_{\mu_{2^d-1}}^0)^{tr}$ as its first column is given by de Boor, Höllig, and Riemenschneider [7, p. 131]: For $0 \neq w \in \mathbf{C}^{2^d}$ let $\mathscr{H}(w)$ be the Householder matrix defined by

$$\mathscr{H}(w) := I - \frac{2}{\|w\|^2} \begin{pmatrix} w_0 \\ \vdots \\ w_{\mu_{2^d-1}} \end{pmatrix} (\overline{w_0}, ..., \overline{w_{\mu_{2^d-1}}}).$$

They show that the matrix defined by

$$\mathscr{P}(v) := \mathscr{H}(P(v) + \sigma(p_0^0)\, e_0) \begin{pmatrix} -\sigma(p_0^0) & 0 & \cdots & 0 \\ 0 & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ 0 & \mathbf{0} & & 1 \end{pmatrix} \tag{5.16}$$

satisfies the above requirements, where e_0 is the first unit vector in $\mathbf{C}^{2^d}$ and

$$\sigma(z) := \begin{cases} z/|z|, & \text{if} \quad z \in \mathbf{C} \backslash \{0\} \\ 1, & \text{if} \quad z = 0. \end{cases}$$

As mentioned above, if $\hat{\varphi}$ is real and nonnegative, then $p_0^0(v) > 0$ for all $v \in \mathbf{R}^d$; therefore in this case $\sigma(p_0^0) \equiv 1$, and it is easily checked that every entry of the matrix $\mathscr{P}$ given by (5.16) is of the form $p_\mu^v = F_\mu^v \circ P$, with F_μ^v a complex valued function real analytic on an open neighbourhood of K, hence $p_\mu^\nu \in A_r$ for all μ, $\nu \in E$, and the corresponding wavelet set belongs to $\mathscr{L}_r^p(\mathbf{R}^d)$.

In view of the orthogonalization procedure in Theorem 4.1 it would of course be sufficient to find a prewavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^p(\mathbf{R}^d)$, $2 \leqslant p \leqslant \infty$. S. D. Riemenschneider and Z. Shen [38] show that if $d \leqslant 3$, $\eta: E \mapsto E$ is a permutation satisfying (5.12) and φ is a (not necessarily orthonormal) scaling function skew-symmetric about some $c_\varphi \in 1/2\mathbf{Z}^d$, then choosing

$$q^\nu := e^{i\eta(\nu)\cdot}[\![\varphi, \varphi]\!](\cdot + \pi\nu)\begin{cases} q^0(\cdot + \pi\nu), & \text{if } 2\nu c_\varphi \text{ is even,} \\ \overline{q^0(\cdot + \pi\nu)}, & \text{if } 2\nu c_\varphi \text{ is odd,} \end{cases} \qquad \nu \in E^*, \quad (5.17)$$

the functions $\{\psi_\nu\}_{\nu \in E^*}$ given by (5.3) form a prewavelet set. Note that this is basically the same construction as (5.13), since $[\![\varphi, \varphi]\!] \equiv 1$ if φ has orthonormal integer shifts by (2.4). As in the orthonormal case, the resulting prewavelets are skew-symmetric or skew-antisymmetric, and they have compact support if φ is compactly supported. By arguments similar to the orthonormal case but involving the scaling function dual to φ one can show that $\varphi \in \mathscr{L}_r^p(\mathbf{R}^d)$ for some $r \in \mathbf{N}_0$, $2 \leqslant p \leqslant \infty$ implies $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^p(\mathbf{R}^d)$. We will not go into the details here.

A construction similar to (5.17) but using a different permutation η is given by C. K. Chui, J. Stöckler and J. D. Ward [15].

6. STRANG-FIX CONDITION VERSUS VANISHING MOMENTS

In this section the proof of Theorem 1.1 will be established.

Proof of Theorem 1.1. Let $r \in \mathbf{N}$. We will first establish the implications (1.9) $\Rightarrow$ (1.10) $\Rightarrow$ (1.11) $\Rightarrow$ (1.9). Suppose ϑ satisfies the Strang-Fix condition of order r. Then

$$D^\alpha \hat{\vartheta}(2j\pi) = 0, \qquad j \in \mathbf{Z}^d \backslash \{0\}, \qquad |\alpha| < r.$$

By Theorem 4.1 there exists an orthonormal scaling function $\varphi \in \mathscr{L}_r^\infty(\mathbf{R}^d)$ and a function $\xi \in A_r \subset C_{2\pi}^r$ such that $\hat{\varphi} = \xi \hat{\vartheta}$. By the Leibniz rule it follows that

$$D^\alpha \hat{\varphi}(2j\pi) = 0, \qquad j \in \mathbf{Z}^d \backslash \{0\}, \qquad |\alpha| < r, \tag{6.1}$$

so φ satisfies the Strang-Fix condition of order r by (2.6), and (1.10) follows.

Now suppose (1.10) is valid; let $\varphi \in \mathscr{L}_r^\infty(\mathbf{R}^d)$ be an orthonormal scaling function satisfying the Strang-Fix condition of order r, let $f \in V_0 \cap \mathscr{L}_r^\infty(\mathbf{R}^d)$ be arbitrary. By the orthonormality of the integer shifts of φ we have

$$f = \sum_{j \in \mathbf{Z}^d} \langle f, \varphi(\cdot - j) \rangle \, \varphi(\cdot - j).$$

Taking Fourier transforms on both sides yields

$$\hat{f} = \sum_{j \in \mathbf{Z}^d} \langle f, \varphi(\cdot - j) \rangle \, e^{-ij \cdot} \hat{\varphi} = [\![f, \varphi]\!] \, \hat{\varphi}.$$

But $[\![f, \varphi]\!] \in A_r \subset C_{2\pi}^r$ by Lemma 4.1, hence by the Leibniz rule f satisfies (1.11). In view of (2.6), the implication (1.11) $\Rightarrow$ (1.9) is obvious.

Let us now show the equivalence of (1.10) and (1.13). From Theorem 4.1 we know that there exists an orthonormal scaling function $\varphi \in \mathscr{L}_r^\infty(\mathbf{R}^d)$, so that Theorem 5.1 ensures the existence of a wavelet set $\{\psi_v\}_{v \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$. To prove the equivalence of (1.10) and (1.13) we will show that φ satisfies the Strang-Fix condition of order r if and only if $\{\psi_v\}_{v \in E^*}$ satisfies the vanishing moment condition of order r.

Observe first that $\{\psi_v\}_{v \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$ implies $\hat{\psi}_v \in C_B^r(\mathbf{R}^d)$, $v \in E^*$, so the vanishing moment condition

$$D^\alpha \hat{\psi}_v(0) = 0, \qquad |\alpha| < r, \quad v \in E^*, \tag{6.2}$$

is equivalent to

$$\| \cdot \|^{-r} \, |\hat{\psi}_v| \in L^\infty([-\pi, \pi]^d), \qquad v \in E^*. \tag{6.3}$$

By Corollary 3.1, (6.3) in turn is equivalent to the existence of a constant $K > 0$ such that

$$E(f, V_n) \leqslant K 2^{-nr} \, \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad n \in \mathbf{Z}. \tag{6.4}$$

So we need to establish the equivalence of (6.4) and (6.1). Suppose (6.4) is valid. Now φ is an orthonormal scaling function, so by (2.4) $[\![\varphi, \varphi]\!] \equiv 1$, and by Corollary 3.1,

$$\frac{((2\pi)^d \sum_{j \in \mathbf{Z}^d,\, j \neq 0} |\hat{\varphi}(\cdot - 2\pi j)|^2)^{1/2}}{\| \cdot \|^r} = \frac{([\![\varphi, \varphi]\!] - (2\pi)^d \, |\hat{\varphi}|^2)^{1/2}}{\| \cdot \|^r} \in L^\infty([-\pi, \pi]^d). \tag{6.5}$$

By continuity this implies that $\hat{\varphi}$ has a zero of order r at $2\pi j$ for any $j \in \mathbf{Z}^d \backslash \{0\}$, or (6.1). By (2.6), φ satisfies the Strang-Fix condition of order r.

Unfortunately, (6.1) does not as easily imply (6.5), so Corollary 3.1 is of no help for the converse implication. This, however, is a classical problem that has been studied by many authors under various assumptions (see e.g. [42, 45, 16, 17, 8, 3, 35, 27, 31, 9], among many others). Best adapted to our situation is the following result by J. Lei [30], which we state here in a slightly modified form:

Result 6.1 [30]. *Let* $r \in \mathbf{N}$, *suppose* $\varphi \in \mathscr{L}_r^\infty(\mathbf{R}^d)$ *is an orthonormal scaling function satisfying the Strang-Fix condition of order r. There exists a constant* $K > 0$ *such that*

$$\|f - P_\varphi^{2^{-n}} f\|_2 \leqslant K 2^{-nr} \max_{|\alpha| = r} \|D^\alpha f\|_2, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad n \in \mathbf{Z}.$$

Lei actually considers $L^p(\mathbf{R}^d)$-approximation for $1 \leqslant p \leqslant \infty$. The above result is in fact established in the proof of [30, Theorem 2.4]; rather than the Strang-Fix condition, J. Lei assumes $\varphi \in \mathscr{W}_{r-1}^\infty(\mathbf{R}^d)$ with derivatives $D^\alpha \varphi \in \mathscr{L}_{r-1}^\infty$, $|\alpha| < r$, and then proceeds to deduce the Strang-Fix condition of order r.

By Result 6.1, assuming that φ satisfies the Strang-Fix condition of order r (6.1), then

$$E(f, V_n) \leqslant K 2^{-nr} \max_{|\alpha| = r} \|D^\alpha f\|_2, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d), \quad n \in \mathbf{Z}.$$

But

$$\max_{|\alpha| = r} \|D^\alpha f\|_2 = \max_{|\alpha| = r} \|\cdot^\alpha \hat{f}\|_2 \leqslant \|f\|_{r,2}, \qquad f \in \mathscr{W}_r^2(\mathbf{R}^d),$$

hence (6.4) holds. This completes the proof of (1.10) $\Leftrightarrow$ (1.13).

It remains to verify (1.13) $\Rightarrow$ (1.14) $\Rightarrow$ (1.12) $\Rightarrow$ (1.13). Suppose (1.13) is valid, i.e., there exists a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}_r^\infty(\mathbf{R}^d)$ satisfying the vanishing moment condition; let $f \in W_0 \cap \mathscr{L}_r^\infty(\mathbf{R}^d)$. By orthonormality, f admits the expansion

$$f = \sum_{\nu \in E^*} \sum_{j \in \mathbf{Z}^d} \langle f, \psi_\nu(\cdot - j) \rangle \, \psi_\nu(\cdot - j)$$

or, taking Fourier transforms on both sides,

$$\hat{f} = \sum_{\nu \in E^*} \sum_{j \in \mathbf{Z}^d} \langle f, \psi_\nu(\cdot - j) \rangle \, e^{-ij\cdot} \hat{\psi}_\nu = \sum_{\nu \in E^*} [\![f, \psi_\nu]\!] \, \hat{\psi}_\nu.$$

But $[\![f, \psi_\nu]\!] \in A_r \subset C^r_{2\pi}$, $\nu \in E^*$, by Lemma 4.1, so using the Leibniz rule we conclude

$$D^\alpha \hat{f}(0) = 0, \qquad |\alpha| < r. \tag{6.6}$$

Furthermore, $f \in W_0 \subset V_1$, hence $f(\cdot/2) \in V_0$. We have already established the equivalence of (1.9), (1.10), (1.11) and (1.13), therefore by (1.11)

$$0 = D^\alpha \left[f\left(\frac{\cdot}{2}\right)\right]^\wedge (2j\pi) = 2^d\, D^\alpha \hat{f}(4j\pi), \qquad j \in \mathbf{Z}^d \backslash \{0\}, \qquad |\alpha| < r. \tag{6.7}$$

Combining this with (6.6), we obtain (1.14).

The implication (1.14) $\Rightarrow$ (1.12) is trivial, since we knew beforehand that there exists a wavelet set $\{\psi_\nu\}_{\nu \in E^*} \subset \mathscr{L}^\infty_r(\mathbf{R}^d)$, and this is in particular a prewavelet set. Finally, (1.12) implies (1.13) by Theorem 4.1 and the Leibniz rule. ∎

7. SOME REMARKS CONCERNING DECAY ASSUMPTIONS

In Theorem 1.1, to prove the equivalence of (1.9)–(1.11) or (1.12)–(1.14), respectively, we only need the Fourier transform of the functions involved to be $(r-1)$ times continuously differentiable, so in view of the results in Section 4 we might replace $\mathscr{L}^\infty_r(\mathbf{R}^d)$ by $\mathscr{L}^\infty_{r-1}(\mathbf{R}^d)$. The only place where the higher order decay is needed is when establishing the equivalence of (1.11) and (1.13). More precisely, the difficulty lies in the use of Result 6.1 and in the fact that a relationship between (3.13) and the Strang-Fix condition, or (3.16) and the vanishing moment condition, can only be established if the corresponding Fourier transforms are in $C^r_B(\mathbf{R}^d)$. Thus our method of proof does not allow us to reduce the decay rate r of the functions considered.

In some instances, however, this can be done using a different method. For example, in the univariate case, the standard approach is to use the explicit representation

$$\hat{\psi}(v) = 1/2 q^1(v/2)\, \hat{\varphi}(v/2) = 1/2 e^{-iv/2}\, \overline{q^0(v/2 + \pi)}\, \hat{\varphi}(v/2), \qquad v \in \mathbf{R}, \tag{7.1}$$

(compare Section 5), where φ is an orthonormal scaling function and q^0 is defined by the relation

$$\hat{\varphi}(v) = 1/2 q^0(v/2)\, \hat{\varphi}(v/2), \qquad v \in \mathbf{R}, \tag{7.2}$$

or

$$q^0(v) = \sum_{j \in \mathbf{Z}^d} 2\langle \varphi, \varphi(2\cdot - j)\rangle\, e^{-ijv}$$
$$= 2[\![\varphi, \varphi(2\cdot)]\!](2v) + e^{-iv}2[\![\varphi, \varphi(2\cdot - 1)]\!](2v), \qquad v \in \mathbf{R}.$$

Thus by Lemma 4.1, if $\varphi \in \mathscr{L}^\infty_{r-1}(\mathbf{R})$, then $q^0 \in A_{r-1} \subset C_B^{r-1}(\mathbf{R})$. Using only the Leibniz rule and the representation (7.2), one can show that φ satisfies the Strang-Fix condition of order r if and only if

$$D^\alpha q^0(\pi) = 0, \qquad \alpha \in \mathbf{N}_0, \quad 0 \leqslant \alpha < r. \tag{7.3}$$

Using again only the Leibniz rule and the fact that $\hat{\varphi}(0) \neq 0$ (which is true for any scaling function in $\mathscr{L}^2(\mathbf{R})$, see [28]), one concludes from (7.1) that (7.3) is equivalent to the vanishing moment condition of order r for the wavelet ψ. Several authors have used this approach to prove at least some of the implications involved under various assumptions ([18, 36, 44]; a complete treatment under very weak assumptions is given in [10].

In higher dimensions the equivalence of (7.3) and the Strang-Fix condition upon φ is still valid; of course (7.3) must now be replaced by

$$D^\alpha q^0(v\pi) = 0, \qquad 0 \leqslant |\alpha| < r, \quad v \in E^*. \tag{7.4}$$

The equivalence of (7.4) and the vanishing moment condition of order r for a wavelet set, however, can only be established by this method if a wavelet set with a representation of type (7.1) is known. Recall from Section 5 that in case φ is a skew-symmetric orthonormal scaling function and dimension $d \leqslant 3$, such a wavelet set can in fact be constructed. Thus in this case and in the univariate case, $\mathscr{L}^\infty_r(\mathbf{R}^d)$ can be replaced by $\mathscr{L}^\infty_{r-1}(\mathbf{R}^d)$ in Theorem 1.1. The general case, however, cannot be treated in this way. There does exist a more direct method to show at least that (1.11) implies (1.13), which does not utilize any results from approximation theory. It is used e.g. by Y. Meyer [37, p. 93] in the instance of a scaling function with rapid decay. This method also works under weaker decay assumptions, but one does need $\hat{\varphi} \in C^r_B(\mathbf{R}^d)$, so it would not help us to reduce the decay rate r.

We would also like to point out that every statement in this paper remains true, if in the definition of $\mathscr{L}^\infty_r(\mathbf{R}^d)$ in (1.2) we would replace "boundedness of $\sum_{j \in \mathbf{Z}^d} |f(x-j)|$ almost everywhere on $\mathbf{R}^d$" by "boundedness on all of $\mathbf{R}^d$." This may be of interest when dealing with continuous functions.

ACKNOWLEDGMENTS

The author thanks Professor W. Dahmen for valuable discussions, as well as Professor P. L. Butzer (both of RWTH Aachen) for helpful comments on the present paper.

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